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BIRZEIT UNIVERSITY

MATHEMATICS DEPARTMENT
MATH330 -Midterm Exam-
Spring 2014/2015

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Q1) Consider the function $f(x) = 3x^2 - e^x - 1$. Estimate the root of $f(x)$ in $[1, 2]$ using the bisection method. Find the first three iterations.

$f(1) = -0.7182 < 0$

$f(2) = 3.6 > 0$

$\Rightarrow [\overset{\ominus}{1}, \overset{\oplus}{2}]$

$\boxed{c_0 = \frac{1+2}{2} = 1.5}$

$f(1.5) = 1.268 > 0$

$\Rightarrow [a_1, b_1] = [\overset{\ominus}{1}, \overset{\oplus}{1.5}]$

$\boxed{c_1 = \frac{1+1.5}{2} = 1.25}$

$f(1.25) = 0.197 > 0$

$\Rightarrow [a_2, b_2] = [\overset{\ominus}{1}, \overset{\oplus}{1.25}]$

$\boxed{c_2 = \frac{1+1.25}{2} = 1.125}$

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Q2) For $f(x) = 3x^2 - e^x - 1$. Estimate the root of $f(x)$ in $[1, 2]$ using the false position method. Find the first two iterations.

$f(1) = -0.71828183$

$f(2) = 3.610943901$

~~$c_0 = b_0 - f(1$~~

$[a_0, b_0] = [\overset{\ominus}{1}, \overset{\oplus}{2}]$

$c_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)} = 2 - \frac{f(2)(2-1)}{f(2) - f(1)} = 1.165914617$

$f(c_0) = -0.1307857944 < 0$

$\Rightarrow [a_1, b_1] = [\overset{\ominus}{1.165914617}, \overset{\oplus}{2}]$

$\Rightarrow c_1 = b_1 - \frac{f(b_1)(b_1 - a_1)}{f(b_1) - f(a_1)} = 2 - \frac{f(b_1)(2 - c_0)}{f(2) - f(c_0)} = 1.195668639$

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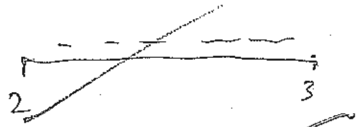
Q3) Let $g(x) = \dfrac{1}{x^3} + 2$. Show that $g(x)$ has a fixed point in the interval $[2, 3]$.

$g(x)$ is continuous in $[2,3]$ خش ا صفار مقام على الفترة $[2,3]$

$g'(x) = \dfrac{-3}{x^{4}} \neq 0 \Rightarrow$  sign of $g'(x)$

$\Rightarrow g(x)$ is always decresing on $[2,3]$

$\Rightarrow$ max is $g(2) = 2.125$

min is $g(3) = 2.037037037$

$[2.037037037, 2.125] \subseteq [2,3]$

so is has fixed pts in $[2,3]$

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Q4) For the above function $g(x)$, if $P_{n+1} = g(P_n)$, show why this iteration converges for any $P_0 \in [2, 3]$

$|g'(x)| = \left|\dfrac{-3}{x^4}\right| = \dfrac{3}{x^4}$

its always decreasing because x^4 increas as x increas

$\Rightarrow$ max $|g'(x)|$ is at $x = 2$

$\Rightarrow$ max$(|g'(x)|) = |g'(2)| = 0.1875$

$|g'(x)| \leq k < 1$

$|g'(x)| < 0.1875 < 1$

So its converge for any $P_0 \in [2,3]$

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Q5) Let $f(x) = 3x^2 - e^x - 1$. Estimate the root of $f(x)$ using Newton's method with $P_0 = 1.5$. Find at least three iterations.

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{3x^2 - e^x - 1}{6x - e^x}$$

$$\Rightarrow P_{n+1} = P_n - \frac{3(P_n)^2 - e^{P_n} - 1}{6P_n - e^{P_n}}$$

A $P_0 = 1.5$

B $P_1 = g(P_0) = g(1.5) = 1.219295341$

C $P_2 = g(P_1) = 1.200154615$

D $P_3 = g(P_2) = 1.200030141$

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Q6) Use the iterations you found in the previous question to find the order of convergence both theoretically and numerically.

$P \simeq P_3 = 1.200030141$

for $f(P) = 0$

$f'(P) = 3.87 \neq 0 \Rightarrow$ simple root $\Rightarrow \boxed{R=2}$

$$A = \left|\frac{f''(P)}{2f'(P)}\right| = \left|\frac{6 - e^P}{2(6P - e^P)}\right| = 0.3453360791 \quad (\text{theoritically})$$

Numerically

$\lvert E_n \rvert = P - P_n$	$\frac{\lvert E_{n+1} \rvert}{\lvert E_n \rvert^2}$
0.2999698593	0.2141006790
0.0192652003	0.3353775529 $\Rightarrow A$
0.000124474	

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Q9) Using 4–digit and rounding, solve the following system using Gaussian elimination (without pivoting).

$$\begin{aligned} 1.231x_1 + 6.342x_2 &= 7.573 \\ 28.34x_1 - 1.120x_2 &= 27.13 \end{aligned}$$

$M_{21} = \dfrac{28.34}{1.231} = 23.02$

$$\left[\begin{array}{cc|c} 1.231 & 6.342 & 7.573 \\ 28.34 & -1.120 & 27.13 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1.231 & 6.342 & 7.573 \\ 0 & 147.1 & 147.2 \end{array}\right]$$

$-1.120 - 23.02 * 6.342 = -147.1$

$27.13 - 23.02 * 7.573 = -147.2$

$\boxed{X_2 = \dfrac{147.2}{147.1} = 1.001}$

$X_1 = \dfrac{7.573 - 6.342 * 1.001}{1.231} = \boxed{0.9951}$

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Q10) A small business has weekly cost of $c(x) = x^4 - x^3 - 2x^2 - 6x + 1500$ where x is the number of hundreds of units produced each week. Approximate (with error less than 10^{-2}) the level of production that yields the minimum cost. **Hint**: minimum cost means $c'(x) = 0$

$c'(x) = 4x^3 - 3x^2 - 4x - 6 = 0$

take $x = \sqrt[3]{\dfrac{3x^2 + 4x + 6}{4}} \Rightarrow P_{n+1} = \sqrt[3]{\dfrac{3x^2 + 4x + 6}{4}}$

~~take $P_0 = 1$~~

~~$\Rightarrow P_1 = 0.5878336719$~~

~~$P_2 = 0.5273385864$~~

~~$P_3 = 0.5189395786 \Rightarrow |P_3 - P_2| = 0.8449607806 * 10^{-2} < 10^{-2}$~~

~~$P_4 = 0.5177703897$~~

~~$\Rightarrow P \simeq 0.5189395786$~~

take $P_0 = 2$

$\Rightarrow P_1 = 1.866255578$

$P_2 = 1.814941271$

$P_3 = 1.79519812$

$P_4 = 1.787594774 \Rightarrow |P_4 - P_3|_5 = 0.7603342136 < 10^{-2} \Rightarrow$ stop

$\Rightarrow \boxed{P \simeq 1.787594774}$

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Q7) Show that if $g(P) = P$ and $g'(P) = g''(P) = 0$, then the sequence generated by $P_{n+1} = g(P_n)$ will converge at least cubically.

taylor series for $g(x)$ around $X = p$

$$\Rightarrow g(x) = g(p) + \cancel{g'(p)(x-p)} + \cancel{\frac{g''(p)(x-p)^2}{2!}} + \frac{g'''(p)(x-p)^3}{3!}$$

$$\Rightarrow \underset{P_{n+1}}{g(x)} = \underset{p}{g(p)} + \frac{g'''(c)(x-p)^3}{3!}$$

$$\Rightarrow |P_{n+1} - P| = \left|\frac{g'''(c)(x-p)^3}{3!}\right| \Rightarrow |P - P_{n+1}| = |C|\,|P - P_n|^3 \quad ; C \text{ is constant}$$

$$\Rightarrow |P - P_{n+1}| = A|P - P_n|^3 \Rightarrow E_{n+1} = A|P - P_n|^3$$

$\Rightarrow$ it will converge cubically

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Q8) Find the cost of solving a 4×4 linear system $Ax = b$ using Cramer's Rule. **Hint**: Cost of the determinant of a 3×3 matrix = 14

Cost of determinant of 4×4 system $= n! \sum_{k=1}^{k=n-1} \frac{1}{k!} + n! - 1$

$$= 4! \sum_{k=1}^{k=3} \frac{1}{k!} + 4! - 1 = 63$$

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$$\left| \; \right| = 4 * \text{cost}\,|A_{3\times3}| + \overset{\text{ضرب}}{4} + \overset{\text{جمع}}{3} = 4(14) + 4 + 3 = 63$$

Cremars Rule needs 5 determinants and 4 division

$$\Rightarrow \text{Cost} = 5 * \text{cost}(|A_{4\times4}|) + 4 = 5 * 63 + 4 = 319$$

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